

Exotic galilean symmetry and non-commutative mechanics in mathematical & in condensed matter physics ^{*}

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Abstract

The “exotic” particle model with non-commuting position coordinates, associated with the two-parameter central extension of the planar Galilei group, can be used to derive the ground states of the Fractional Quantum Hall Effect. The relation to other NC models is discussed. Anomalous coupling is presented. Similar equations arise for a semiclassical Bloch electron, used to explain the anomalous/spin/optical Hall effects.

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1 Introduction : “Exotic” Galilean symmetry

Central extensions first entered physics when Heisenberg realized that, in the quantum mechanics of a massive particle, the position and momentum operators did not commute. As a consequence, the group of space translations only acts up-to-phase on the quantum Hilbert space. Expressed in

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more mathematical terms, it is not the [commutative] translation group itself, only its [non-commutative] 1-parameter central extension is represented unitarily.

Similarly, for a massive non-relativistic system Galilean boosts only act up-to phase, so that it is its 1-parameter central extension that acts unitarily. True representations only arise for massless particles.

Are there further extension parameters ? The question has been asked and solved by Bargmann [1] : in $d \geq 3$ space dimensions, the Galilei group only admits a 1-parameter central extension identified with physical mass, m . Lévy-Leblond [2] has recognized, however that, owing to the commutativity of the planar rotation group $O(2)$, the Galilei group in the plane admits a second “exotic” extension, highlighted by the non-commutativity of Galilean boost generators,

$$[K_1, K_2] = i\kappa, \quad (1)$$

where κ is the new extension parameter. This fact has long been considered, however, a mere mathematical curiosity, as planar physics has been viewed itself as a toy. Around 1995 the situation started to change, though, with the construction of physical models with such an “exotic” structure [3, 4]. These models have the strange feature that the Poisson bracket of the planar coordinates does not vanish,

$$\{x_1, x_2\} = \frac{\kappa}{m^2} \equiv \theta. \quad (2)$$

Physical consequences, drawn in Ref. ([5]) are presented in Section 2 below.

Independently and around the same time, similar structures were considered in condensed matter physics, namely for the Bloch electron [6], where it has been argued that the semiclassical dynamics should involve a Berry curvature term which induces an “anomalous” velocity term of the same form as in the “exotic” model of Ref. ([5]). Recent developments include the Anomalous [7], the Spin [8] and the Optical [9, 10, 11] Hall effects.

Below we review the exotic point-particle model of Ref. ([5]), followed by a brief outline of the semiclassical Bloch electron. Let us note, in conclusion, that exotic Galilean symmetry can also been extended to non-commutative (Moyal) field theory [12, 13].

2 “Exotic” mechanics in the plane

Our present understanding of the Fractional Quantum Hall Effect is based on the motion of charged vortices in a magnetic field [14]. Such vortices arise as exact solutions in a field theory of matter coupled to an abelian gauge field A_ν , whose dynamics is governed by the Chern-Simons term [15, 16]. Theory can be either relativistic or nonrelativistic. For the latter, boosts commute, but exotic Galilean symmetry can be found in a Moyal-version of Chern-Simons field-theory [13].

In Ref. ([3, 5]) Souriau’s “orbit method” [17] was used to construct a classical system associated with Lévy-Leblond’s “exotic” Galilean symmetry. It has an “exotic” symplectic form and a free Hamiltonian,

$$\Omega_0 = dp_i \wedge dq^i + \frac{1}{2}\theta \varepsilon_{ij} dp^i \wedge dp^j, \quad (3)$$

$$H_0 = \frac{\vec{p}^2}{2m}. \quad (4)$$

The associated free motions follow the usual straight lines; the “exotic” structure only enters the conserved quantities, namely the boost and the angular momentum,

$$\begin{aligned} j &= \varepsilon_{ij} x_i p_j + \frac{\theta}{2} \vec{p}^2, \\ K_i &= m x_i - p_i t + m\theta \varepsilon_{ij} p_j. \end{aligned} \quad (5)$$

The “exotic” structure behaves hence roughly as spin: it contributes to some conserved quantities, but the new terms are separately conserved. The new structure does not seem to lead to any new physics.

The situation changes dramatically, though, if the particle is coupled to a gauge field. Applying Souriau’s prescription [17] yields indeed

$$\Omega = \Omega_0 + eB dq_1 \wedge dq_2, \quad H = H_0 + eV. \quad (6)$$

The associated Poisson bracket then automatically satisfies the Jacobi identity. The resulting equations of motion read

$$\begin{aligned} m^* \dot{x}_i &= p_i - em\theta \varepsilon_{ij} E_j, \\ \dot{p}_i &= eE_i + eB \varepsilon_{ij} \dot{x}_j, \end{aligned} \quad (7)$$

where $\theta = k/m^2$ is the non-commutative parameter and we have introduced the *effective mass* m^*

$$m^* = m(1 - e\theta B). \quad (8)$$

The novel features, crucial for physical applications, are two-fold : Firstly, the relation between velocity and momentum, (37), contains an “anomalous velocity” term, so that \dot{x}_i and p_i are not in general parallel. The second one is the interplay between the exotic structure and the magnetic field, yielding the effective mass m^* in (38).

Equations (7) come from the Lagrangian

$$\int (\mathbf{p} - \mathbf{A}) \cdot d\mathbf{x} - \frac{p^2}{2} dt + \frac{\theta}{2} \mathbf{p} \times d\mathbf{p}. \quad (9)$$

When $m^* \neq 0$, (7) is also a Hamiltonian system, $\dot{\xi} = \{h, \xi^\alpha\}$, with $\xi = (p_i, x^j)$ and Poisson brackets

$$\begin{aligned} \{x_1, x_2\} &= \frac{m}{m^*} \theta, \\ \{x_i, p_j\} &= \frac{m}{m^*} \delta_{ij}, \\ \{p_1, p_2\} &= \frac{m}{m^*} eB. \end{aligned} \quad (10)$$

A remarkable property is that for *vanishing effective mass* $m^* = 0$, i.e., when the magnetic field takes the critical value

$$B = \frac{1}{e\theta}, \quad (11)$$

the system becomes singular. Then “Faddeev-Jackiw” (alias symplectic) reduction yields an essentially two-dimensional, simple system, reminiscent of “Chern-Simons mechanics” [18]. The symplectic plane plays, simultaneously, the role of both configuration and phase space. The only motions are those which follow a generalized Hall law; quantization of the reduced system yields the “Laughlin” wave functions [14], which are in fact the ground states in the Fractional Quantum Hall Effect (FQHE).

The relations (10) diverge as $m^* \rightarrow 0$, but after reduction we get, cf.(2),

$$\{x_1, x_2\} = \frac{1}{eB} = \theta. \quad (12)$$

3 Relation to another non-commutative mechanics

The exotic relations (10) are similar to those proposed in Ref. [21],

$$\begin{aligned}\{x_i, x_j\} &= \theta \epsilon_{ij}, \\ \{x_i, p_j\} &= \delta_{ij}, \\ \{p_1, p_2\} &= eB.\end{aligned}\tag{13}$$

The eqns of motion $\dot{\xi} = \{\xi, H\}$, where $\xi = (p_i, x^j)$ and $H = \frac{p^2}{2m} + eV(x)$ is the standard Hamiltonian, read

$$\begin{aligned}mx'_i &= p_i - em\theta\epsilon_{ij}E_j, \\ p'_i &= eB\epsilon_{ij}\frac{p_j}{m} + eE_i,\end{aligned}\tag{14}$$

where we noted “time” by T ; $(\cdot)' = \frac{d}{dT}$. Then a short calculation shows that

$$\{x_i, \{p_1, p_2\}\}_{cycl} = e\theta\epsilon_{ij}\partial_j B,\tag{15}$$

so that the Jacobi identity is only satisfied if $B = \text{const.}$

How can this theory be extended to an arbitrary B ?

• Let us first assume that $B = \text{const.}$ s.t. $m^* \neq 0$, and *let us redefine the time*¹, as

$$T \rightarrow t = (1 - e\theta B)T \quad \Rightarrow \quad \frac{d}{dT} = (1 - e\theta B) \frac{d}{dt}.\tag{16}$$

Then eqns. (14) carried into the exotic eqns (7). It follows that the two theories are, under these conditions, equivalent.

• The crucial fact is that the time redefinition actually extends the previous theory, since it carries into the “exotic model”, where the Jacobi identity holds for any, not necessarily constant B .

The model (13) has another strange feature. Let us indeed assume that the magnetic field is radially symmetric, $B = B(r)$. One would then expect to have conserved angular momentum. For constant B , applying Noether’s theorem to an infinitesimal rotation $\delta\xi_i = \epsilon_{ij}\xi_j$ yields indeed $\delta\xi_i = -\{J^{NP}, \xi_i\}$, with

$$J^{NP} = \frac{1}{1 - e\theta B} \underbrace{\left(\vec{q} \times \vec{p} + \frac{\theta}{2} \vec{p}^2 + \frac{eB}{2} \vec{q}^2 \right)}_j.\tag{17}$$

¹This was suggested to me by G. Marmo.

This differs from the standard expression in the pre-factor

$$\frac{1}{1 - e\theta B}.$$

For $B = B(r) \neq \text{const.}$, however, (17) is not in general conserved :

$$\frac{dJ^{NP}}{dT} = \frac{e\theta}{1 - e\theta B} \partial_i B x'_i,$$

while j in (5) is conserved.

4 Physical origin of exotic structure

A free relativistic “elementary” particle in the plane corresponds to a unitary representation of the planar Lorentz group $o(2, 1)$. According to geometric quantization, these representations are associated with the coadjoint orbits of the planar Lorentz group $SO(2, 1)$, endowed with their canonical symplectic structures. Following Souriau, these latters can in turn be viewed as classical phase spaces.

For the planar Lorentz group, the procedure yields [23]

$$\Omega_0 = dp_\alpha \wedge dx^\alpha + \frac{s}{2} \epsilon^{\alpha\beta\gamma} \frac{p_\alpha dp_\beta \wedge dp_\gamma}{(p^2)^{3/2}}, \quad (18)$$

$$H_0 = \frac{1}{2m} (p^2 - m^2 c^2). \quad (19)$$

Then, as pointed out by Jackiw and Nair [22], the free exotic model can be recovered considering a tricky non-relativistic limit, namely

$$s/c^2 \rightarrow \kappa = m^2 \theta. \quad (20)$$

Then $\Omega_0|_{H_0=0}$ goes over into the exotic symplectic form. Intuitively, the exotic structure can be viewed as a “non-relativistic shadow” of relativistic spin.

At the level of the field equations, a similar procedure, applied to the infinite-component Majorana-type equation considered by Jackiw & Nair, or by Plyushchay [23] yields a first-order infinite-component “Lévy-Leblond type” system [24].

The exotic Galilei group can itself be derived from the planar Poincaré group by “Jackiw-Nair” contraction [22]. One starts with the planar Lorentz generators,

$$\{J^\alpha, J^\beta\} = \epsilon^{\alpha\beta\gamma} J_\gamma. \quad (21)$$

For the classical system

$$J_\mu = \epsilon_{\mu\nu\rho} x^\nu p^\rho + s \frac{p_\mu}{\sqrt{p^2}}. \quad (22)$$

Non-relativistic boost are the “JN” limits of

$$\frac{1}{c} \epsilon_{ij} J^j \rightarrow m x_i - p_i t + m \theta \epsilon_{ij} p_j = K_i, \quad (23)$$

and the exotic relation is recovered,

$$\{K_1, K_2\} = J_0/c^2 \rightarrow \frac{s}{c^2} = \kappa. \quad (24)$$

The angular momentum is in turn

$$J_0 = \vec{x} \times \vec{p} + s + \frac{s}{m^2 c^2} \vec{p}^2 \rightarrow \vec{x} \times \vec{p} + \frac{1}{2} \kappa \vec{p}^2 = j. \quad (25)$$

whereas the divergent term $s = \kappa c^2$ has to be removed by hand.

5 Anyons in e.m. fields

Chou, Nair, Polychronakos [25] suggested to describe an anyon in an electromagnetic field by the equations

$$\begin{aligned} m \frac{dx^\alpha}{d\lambda} &= p^\alpha & (\text{velocity-momentum}) \\ \frac{dp^\alpha}{d\lambda} &= \frac{e}{m} F^{\alpha\beta} p_\beta & (\text{Lorentz equation}) \end{aligned} \quad (26)$$

These equations are Hamiltonian, with symplectic form and Hamilton’s function

$$\Omega = \Omega_0 + \frac{1}{2} e F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \quad (27)$$

$$H = H_0 + \frac{es}{2m\sqrt{p^2}} \epsilon_{\alpha\beta\gamma} F^{\alpha\beta\gamma} p^\gamma, \quad (28)$$

respectively. Let us observe that the second, non-minimal term in the Hamiltonian is dictated by the form of the velocity relation in (26).

As proved by Chou et al. in Ref. ([25]), their model has gyromagnetic ratio $g = 2$, which has long been believed by high-energy-physics theoreticians [25, 26] to be the “correct” g value of anyons. Experimental evidence

shows, however, that in various condensed-matter situations including the Fractional Quantum Hall Effect, the measured value of g is approximately zero[27].

Is it possible to construct an “anomalous” model with $g \neq 2$? The answer is affirmative [28]. Planar spin has to satisfy the relation $S_{\alpha\beta}p^\beta = 0$. The spin tensor has, therefore, the form

$$S_{\alpha\beta} = \frac{s}{\sqrt{p^2}} \epsilon_{\alpha\beta\gamma} p^\gamma. \quad (29)$$

Introducing the shorthand $-F_{\alpha\beta}S^{\alpha\beta} = F \cdot S$, the Hamiltonian (28) is presented as

$$H^{CNP} = \frac{1}{2m} (p^2 - M^2 c^2) \quad \text{where} \quad M^2 = m^2 + \frac{e}{c^2} F \cdot S. \quad (30)$$

Let us observe that the “mass” M depends here on spin-field coupling. Our clue for generalizing this model has been the formula put forward by Duval more than three decades ago [29] : let us posit instead of (30) the mass formula

$$M^2 = m^2 + \frac{g}{2} \frac{e}{c^2} F \cdot S, \quad (31)$$

where g is an arbitrary real constant. Then *consistent equations of motion* are obtained for any g , namely

$$D \frac{dx^\alpha}{d\tau} = G \frac{p^\alpha}{M} + (g-2) \frac{es}{4M^2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \quad (32)$$

$$\frac{dp^\alpha}{d\lambda} = \frac{e}{m} F^{\alpha\beta} p_\beta, \quad (33)$$

where the coefficients denote the complicated, field-dependent expressions

$$D = 1 + \frac{eF \cdot S}{2M^2 c^2}, \quad G = 1 + \frac{g}{2} \frac{eF \cdot S}{2M^2 c^2}. \quad (34)$$

For the choice $g = 2$ the generalized model plainly reduces to eqn. (26) of Chou et al. in ([25]).

We can now consider the “Jackiw-Nair” non-relativistic limit of the above relativistic model. This provides us, for any g , with the Lorentz eqn. (38), supplemented with

$$(M_g D) \dot{x}_i = G p_i - (1 - \frac{g}{2}) e M_g \theta \epsilon_{ij} E_j, \quad (35)$$

where

$$M_g = m(\sqrt{1 - g\theta eB}), \quad D = (1 - (g+1)\theta eB), \quad G = (1 - (3g/2)\theta eB).$$

- It is a most important fact that, for *any* $g \neq 2$, the only consistent motions follow a generalized Hall law, whenever the field takes *either* of the critical values

$$B = \frac{1}{1+g} \frac{1}{e\theta} \quad \text{or} \quad \frac{2}{3g} \frac{1}{e\theta}. \quad (36)$$

One can indeed show that, for any $g \neq 2$, the models can be transformed into each other by a suitable redefinition. For $g = 0$ the equations become identically satisfied. See [28] for details.

- In particular, for $g = 0$ the minimal exotic model of Ref. ([5]) is obtained. The latter is, hence, *not* the NR limit of the model of [25] (26) which has, as said, $g = 2$. The experimental evidence in [27] is, hence, a strong argument in favour of the minimal model of Ref. ([5]).

- $g = 2$ is in fact the only case, when the velocity & the momentum are parallel. This is, however, *not* required by any first principle, as advocated a long time ago : a perfectly consistent model is obtained for any g [29, 30, 31]. Having non-parallel velocity and momentum seems to be unusual in high-energy physics; it is, however, a well accepted and even crucial requirement in condensed matter physics, as explained in the next Section.

6 The semiclassical Bloch electron

Around the same time and with no relation to the above developments, a very similar theory has arisen in solid state physics [6]. Applying a Berry-phase argument to a Bloch electron in a lattice, a semiclassical model can be derived [6]; the equations of motion in the n^{th} band read

$$\dot{\mathbf{r}} = \frac{\partial \epsilon_n(\mathbf{p})}{\partial \mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\Theta}(\mathbf{p}), \quad (37)$$

$$\dot{\mathbf{p}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}), \quad (38)$$

where $\mathbf{r} = (x^i)$ and $\mathbf{p} = (p_j)$ denote the electron's three-dimensional intracell position and quasimomentum, respectively, $\epsilon_n(\mathbf{p})$ is the band energy. The purely momentum-dependent $\boldsymbol{\Theta} = (\Theta_i)$ is the Berry curvature of the electronic Bloch states, $\Theta_i(\mathbf{p}) = \epsilon_{ijl} \partial_{\mathbf{p}_j} a_l(\mathbf{p})$, where a_i is the Berry connection. A non-trivial Berry connection requires broken time-reversal symmetry, as it happens, e. g., in GaAs heterostructures [6].

Recent applications of the model include the Anomalous [7] and the Spin [8] Hall Effects. All these developments are based on the anomalous velocity term

$$\dot{\mathbf{p}} \times \boldsymbol{\Theta}(\mathbf{p}) \quad (39)$$

that corresponds to the anomalous current advocated by Karplus and Luttinger as long as fifty years ago [7] to explain the Anomalous Hall Effect, observed in some ferromagnetic matter in the absence of a magnetic field. Now, as confirmed experimentally by Fang et al. in Ref. ([7]), in the Anomalous Hall Effect the Berry curvature can take the form of a *monopole in \mathbf{p} -space*,

$$\boldsymbol{\Theta} = g \frac{\mathbf{p}}{p^3}, \quad (40)$$

which is indeed the only possibility consistent with spherical symmetry [20]. For $\mathbf{B} = 0$ we have $\dot{\mathbf{p}} = -e\mathbf{E}$. Then, taking the parabolic case $\epsilon_n(\mathbf{p})$ for simplicity, the velocity relation (37) becomes

$$\dot{\mathbf{r}} = \mathbf{p} + \frac{eg}{p^3} \mathbf{E} \times \mathbf{p}. \quad (41)$$

The anomalous term shifts the velocity and deviates, hence, the particle's trajectory perpendicularly to the electric field – just like in the ordinary Hall effect.

Eqns. (37-38) derive from the Lagrangian

$$L^{Bloch} = (p_i - eA_i(\mathbf{r}, t))\dot{x}^i - (\epsilon_n(\mathbf{p}) + eV(\mathbf{r}, t)) + a^i(\mathbf{p})\dot{p}_i, \quad (42)$$

and are also consistent with the Hamiltonian structure [19]

$$\{x^i, x^j\}^{Bloch} = \frac{\epsilon^{ijk}\Theta_k}{1 + e\mathbf{B} \cdot \boldsymbol{\Theta}}, \quad (43)$$

$$\{x^i, p_j\}^{Bloch} = \frac{\delta_j^i + eB^i\Theta_j}{1 + e\mathbf{B} \cdot \boldsymbol{\Theta}}, \quad (44)$$

$$\{p_i, p_j\}^{Bloch} = -\frac{\epsilon_{ijk}eB^k}{1 + e\mathbf{B} \cdot \boldsymbol{\Theta}} \quad (45)$$

and Hamiltonian $h = \epsilon_n + eV$ [20].

Restricted to the plane, these equations reduce to the exotic equations (7), provided $\Theta_i = \theta\delta_{i3}$. For $\epsilon_n(\mathbf{p}) = \mathbf{p}^2/2m$ and choosing $A_i = -(\theta/2)\epsilon_{ij}p_j$, the semiclassical Bloch Lagrangian (42) becomes the “exotic” expression (9).

The exotic galilean symmetry is lost if θ is not constant, though.

A similar pattern arises in optics [9, 10, 11] : to first order in the gradient of the refractive index n , spinning light is approximately described by the equations

$$\dot{\mathbf{r}} \approx \mathbf{p} - \frac{s}{\omega} \text{grad}\left(\frac{1}{n}\right) \times \mathbf{p}, \quad \dot{\mathbf{p}} \approx -n^3\omega^2 \text{grad}\left(\frac{1}{n}\right). \quad (46)$$

where s denotes the photon's spin. Here we recognize once again an anomalous velocity relation of the type (37). The new term makes the light's trajectory deviate from that predicted in ordinary geometrical optics, giving rise to an "optical Magnus effect" [9]. A manifestation of this is the displacement of the light ray perpendicularly to the plane of incidence at the interface of two media with different refraction index : this is the "Optical Hall Effect" [10, 11].

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